Scattering problem in deformed space with minimal length

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March 29, 2007

Abstract

We investigated the elastic scattering problem with deformed Heisenberg algebra leading to the existence of a minimal length. The continuity equations for the moving particle in deformed space were constructed. We obtained the Green's function for a free particle, scattering amplitude and cross-section in deformed space. We also calculated the scattering amplitudes and differential cross-sections for the Yukawa and the Coulomb potentials in the Born approximation.

1 Introduction

In recent years a number of works were devoted to the investigation of the quantum mechanical problems with deformed commutation relations. Such an interest was prompted by several independent lines of investigation in string theory and quantum gravity which proposed the existence of a finite lower bound to a possible resolution of length [1, 2, 3]. Kempf *et al.* argued that the minimal length can be obtained from the deformed (generalized) commutation relations [4, 5, 6, 7, 8]. But it should to note that the deformed algebra leading to a quantized space-time was originally introduced by Snyder in the relativistic case [9]. The deformed algebra leading to the existence of a minimal length in D-dimensional case reads

$$\begin{aligned} &[X_i, P_j] = i\hbar(\delta_{ij}(1+\beta P^2) + \beta' P_i P_j), [P_i, P_j] = 0, \\ &[X_i, X_j] = i\hbar\frac{(2\beta-\beta') + (2\beta+\beta')\beta P^2}{1+\beta P^2}(P_i X_j - P_j X_i), \end{aligned} \tag{1}$$

where β, β' are the parameters of deformation. We assume that these quantities are nonnegative $\beta, \beta' \geqslant 0$. Using the uncertainty relation one can obtain that minimal length equals $\hbar\sqrt{\beta+\beta'}$.

Deformed Heisenberg algebra (1) causes new complications in solving quantum mechanical problems. There just a few known problems for which the energy spectra have been found exactly. They are one-dimensional harmonic oscillator with minimal uncertainty in position [5] and also with minimal uncertainty in position and momentum [10, 11], D-dimensional isotropic harmonic oscillator [12, 13], three-dimensional relativistic Dirac oscillator [14] and one-dimensional Coulomb problem [15].

The hydrogen atom problem has the crucial role for the understanding of the key points of modern physics. So it is interestingly to study this problem when the position and the momenta operators satisfy the deformed commutation relations (1). The Coulomb problem in deformed space with minimal length was firstly considered by Brau in the special case $2\beta = \beta'$ [16]. The general case of deformation $2\beta \neq \beta'$ was examined in Ref. [17]. The authors utilized perturbation theory for calculating of the corrections to the energy levels. But the perturbation theory used by the authors did not allow to obtain the corrections for the s-levels. To avoid this problem the authors used the cut-off procedure and the numerical methods. In our work [18] a modified perturbation theory allowing to calculate corrections for arbitrary energy levels including s-levels was developed. In [19] a modified perturbation theory was used for finding of the corrections to the ns-levels of the hydrogen atom.

It is evidentially that the scattering problem is a key one because it can also manifest noncommutative effects on the experimental level. At the same time as far as we know there were no papers on the investigation of the quantum-mechanical scattering problem in deformed space with minimal length described by algebra (1). There are only a few papers where scattering problem on the noncommutative space with canonical deformation $[X_i, X_j] = i\theta_{ij}$ was considered [20, 21, 22].

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In this work a scattering problem in the deformed space with minimal length is studied. We consider the elastic scattering on Yukawa and Coulomb potentials. This paper is organized as follows. In the second section we construct the continuity equation for the moving particle in the deformed space. In the third section we obtain the expressions for the scattering amplitude and the differential cross-section. In the fourth section we calculate the scattering amplitude and the cross-section in the Born approximation. And finally the fifth section contains the discussion.

2 Continuity equation

Before considering the scattering problem in deformed space it is necessary to establish continuity equation at first. The construction of the continuity equation in deformed space for the particle moving in the arbitrary external field is more complicated than in the case of ordinary quantum mechanics. Since the main goal of our paper is the scattering problem for a particle in the external Yukawa and Coulomb fields we investigate the continuity equation for these particular problems only.

Let us consider the particle in deformed space described by the Hamiltonian

$$H = \frac{\mathbf{P}^2}{2m} + U(R), \tag{2}$$

where $U(R) = -e^2 \frac{e^{-\lambda R}}{R}$ is the Yukawa potential. The operators of position X_i and momentum P_i obey the deformed commutation relations (1) and $R = \sqrt{\sum_{i=1}^{3} X_i^2}$ is the distance operator. The Coulomb potential can be obtained from the Yukawa potential in the limit $\lambda = 0$.

For constructing the continuity equation we write the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi. \tag{3}$$

One can write the following relation using equation (3)

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}(\psi^* H\psi - \psi H \psi^*), \tag{4}$$

where $\rho = |\psi|^2$ and the Hamiltonian is real: $H^* = H$.

Then for constructing the continuity equation it is necessary to use the representation of the operators of positions and momenta satisfying the deformed commutation relations (1). The momentum representation for such an algebra is well known, but it is not convenient for us. We use the following representation that fulfils algebra (1) in the first order in β, β'

$$\begin{cases} X_i = x_i + \dfrac{2\beta - \beta'}{4}\left(x_i p^2 + p^2 x_i\right), \\ P_i = p_i + \dfrac{\beta'}{2} p_i p^2, \end{cases} \tag{5}$$

where $p^2 = \sum_{j=1}^{3} p_j^2$ and the operators x_i, p_j satisfy the canonical commutation relation. The position representation $x_i = x_i$, $p_j = i\hbar \frac{\partial}{\partial x_j}$ can be taken for the ordinary Heisenberg algebra. We notice that in the special case $2\beta = \beta'$ the position operators commute in linear approximation over the deformation parameters, i.e. $[X_i, X_j] = 0$.

We rewrite Hamiltonian (2) using representation (5) and develop it with respect to β, β' up to the first order. The main problem is related with the expansion of the distance operator R and inverse distance operator $1/R$. In our previous paper [18] for the elimination of divergent term $1/r$ at $r = 0$ in the expansion of R and as a consequence $1/r^3$ in $1/R$ we proposed the so-called shifted expansion. So for R we have [18]

$$R = \sqrt{r^2 + b^2} + \frac{\alpha}{2}(rp^2 + p^2 r), \tag{6}$$

where $\alpha = (2\beta - \beta')/2$ and $b = \hbar\sqrt{2\alpha}$.

Using expansion (6) we can represent decomposition for the inverse distance operator just as it was done in [18]

$$\frac{1}{R} = \frac{1}{\sqrt{r^2 + b^2}} - \frac{\alpha}{2}\left(\frac{1}{r}p^2 + p^2\frac{1}{r}\right). \tag{7}$$

For the expansion of the exponential operator we use the T-exponent representation

$$e^{-\lambda\left[\sqrt{r^2+b^2} + \frac{\alpha}{2}(rp^2 + p^2 r)\right]} = e^{-\lambda\sqrt{r^2+b^2}} T\exp\left(\frac{\alpha}{2}\int_0^\lambda \mathrm{d}\lambda' \mathrm{e}^{\lambda'\sqrt{\mathrm{r}^2+\mathrm{b}^2}}(\mathrm{rp}^2 + \mathrm{p}^2\mathrm{r})\mathrm{e}^{-\lambda'\sqrt{\mathrm{r}^2+\mathrm{b}^2}}\right). \tag{8}$$

In this paper we restrict ourselves to the first order approximation over the deformation parameters. So we develop the T-exponent operator into the series and take into account only the first order terms in α. Then we use the explicit form for the operator $p^2 = -\hbar^2\nabla^2$ and after simple transformations we represent expression (8) in the following way

$$e^{-\lambda\left[\sqrt{r^2+b^2}+\frac{\alpha}{2}(rp^2+p^2r)\right]} = e^{-\lambda\sqrt{r^2+b^2}}\left(1+\frac{\alpha\hbar^2}{2}\int_0^{\lambda} \mathrm{d}\lambda' \left[2\lambda'^2 \mathrm{r} - 2\lambda'((\mathbf{r}\nabla)+(\nabla\mathbf{r})) + \mathrm{r}\nabla^2 + \nabla^2\mathrm{r}\right]\right) \simeq$$
$$e^{-\lambda\sqrt{r^2+b^2}} + \frac{\alpha\hbar^2}{2}e^{-\lambda r}\left(\frac{2}{3}\lambda^3 r - \lambda^2((\mathbf{r}\nabla)+(\nabla\mathbf{r})) + \lambda(r\nabla^2+\nabla^2 r)\right). \quad (9)$$

Using expansion (7) for the inverse distance and decomposition (9) we represent the Yukawa potential in the form of

$$U(R) = -e^2\frac{e^{-\lambda R}}{R} = -e^2\left[\frac{e^{-\lambda\sqrt{r^2+b^2}}}{\sqrt{r^2+b^2}} + \frac{\alpha\hbar^2}{2}e^{-\lambda r}\left(\frac{1}{r}\nabla^2+\nabla^2\frac{1}{r}\right) + \frac{\alpha\hbar^2}{2}\frac{e^{-\lambda r}}{r}\left(\frac{2}{3}\lambda^3 r - \right.\right.$$
$$\left.\left.\lambda^2((\mathbf{r}\nabla)+(\nabla\mathbf{r})) + \lambda(r\nabla^2+\nabla^2 r)\right) + \frac{\alpha\hbar^2}{2}e^{-\lambda r}\left(\frac{2\lambda^2}{r} - 2\lambda\frac{1}{r}((\mathbf{r}\nabla)+(\nabla\mathbf{r}))\frac{1}{r}\right)\right] = U(\mathbf{r},\mathbf{p}) \quad (10)$$

and we note that here $\mathbf{p} = -i\hbar\nabla$.

Thus Hamiltonian (2) in canonical variables reads

$$H = \frac{p^2}{2m} + \frac{\beta' p^4}{2m} + U(\mathbf{r},\mathbf{p}). \quad (11)$$

We substitute Hamiltonian (11) in relation (4) and take into consideration the explicit form for the operator $p^2 = -\hbar^2\nabla^2$. After simple transformations we obtain

$$\frac{\partial\rho}{\partial t} = \frac{1}{i\hbar}\nabla\left[-\frac{\hbar^2}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \frac{\beta'\hbar^4}{2m}(\psi^*\nabla^3\psi - \psi\nabla^3\psi^* - \nabla\psi^*\nabla^2\psi + \nabla\psi\nabla^2\psi^*) - \frac{e^2\hbar^2(2\beta-\beta')}{4}\times\right.$$
$$\left.\left(\psi^*\left[e^{-\lambda r}\left(\frac{1}{r}\nabla+\nabla\frac{1}{r}\right) + \lambda\frac{e^{-\lambda r}}{r}(r\nabla+\nabla r)\right]\psi - \psi\left[e^{-\lambda r}\left(\frac{1}{r}\nabla+\nabla\frac{1}{r}\right) + \lambda\frac{e^{-\lambda r}}{r}(r\nabla+\nabla r)\right]\psi^*\right)\right], \quad (12)$$

where $\nabla^3 = (\nabla,\nabla)\nabla$.

Relation (12) can be represented in the following form

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\mathbf{j} = 0. \quad (13)$$

So we have a well known continuity equation where

$$\mathbf{j} = -\frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \frac{\beta'\hbar^4}{2m}(\psi^*\nabla^3\psi - \psi\nabla^3\psi^* - \nabla\psi^*\nabla^2\psi + \nabla\psi\nabla^2\psi^*) - \frac{e^2\hbar^2(2\beta-\beta')}{4}\times\right.$$
$$\left.\left(\psi^*\left[e^{-\lambda r}\left(\frac{1}{r}\nabla+\nabla\frac{1}{r}\right) + \lambda\frac{e^{-\lambda r}}{r}(r\nabla+\nabla r)\right]\psi - \psi\left[e^{-\lambda r}\left(\frac{1}{r}\nabla+\nabla\frac{1}{r}\right) + \lambda\frac{e^{-\lambda r}}{r}(r\nabla+\nabla r)\right]\psi^*\right)\right] \quad (14)$$

is the density current for the particle in the external Yukawa field.

In a particular case of $\lambda = 0$ we obtain the density current for a particle in the Coulomb field

$$\mathbf{j} = \frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \frac{\beta'\hbar^4}{2m}(\psi^*\nabla^3\psi - \psi\nabla^3\psi^* - \nabla\psi^*\nabla^2\psi + \nabla\psi\nabla^2\psi^*) - \right.$$
$$\left.\frac{\hbar^2 e^2(2\beta-\beta')}{4}\left(\psi^*\left(\frac{1}{r}\nabla+\nabla\frac{1}{r}\right)\psi - \psi\left(\frac{1}{r}\nabla+\nabla\frac{1}{r}\right)\psi^*\right)\right). \quad (15)$$

Expressions (14) and (15) for the density current in the deformed case for a particle moving in the external Yukawa or Coulomb fields are somewhat different from the density current in the ordinary quantum mechanics: we have two additional terms into the continuity equation. One of them is caused by the deformed kinetic energy. The second contribution is caused by the external fields. We notice that in the special case of $2\beta = \beta'$ when the position operators commute in linear approximation over the deformation parameters, i.e. $[X_i, X_j] = 0$, the potential energy does not give any contribution into the continuity equation.

But we want to stress that in the scattering problems we calculate the density current at large distances from the scattering center and we can neglect the terms caused by the field. So the density current for a particle scattered by the Yukawa or the Coulomb fields at large distances from the scattering center takes the same form as for a free particle in the deformed space

$$\mathbf{j} = \frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \frac{\beta'\hbar^4}{2m}(\psi^*\nabla^3\psi - \psi\nabla^3\psi^* - \nabla\psi^*\nabla^2\psi + \nabla\psi\nabla^2\psi^*)\right). \quad (16)$$

3 Scattering amplitude

In this section we examine the scattering of the particle on the arbitrary potential $U(\mathbf{R})$. Position operators in the potential energy operator fulfil the deformed Heisenberg algebra (1). Using representation (5) the potential energy operator can be represented as a function of the canonical variables x_i, p_j: $U(\mathbf{R}) = U(\mathbf{r}, \mathbf{p})$ (see eq. (10)). The explicit form of the potential is taken into account in the final relations only.

Let us consider the Schrödinger equation

$$\left(\frac{p^2}{2m} + \frac{\beta' p^4}{2m} + U(\mathbf{r}, \mathbf{p})\right)\Psi = E\Psi. \tag{17}$$

We suppose that $U(\mathbf{r}, \mathbf{p}) \to 0$ when $r \to \infty$ and at large distances from the scatterer the motion of a particle is free. The kinetic energy of the incident particle equals

$$E = \frac{\hbar^2 k^2}{2m}(1 + \beta' \hbar^2 k^2), \tag{18}$$

where $\mathbf{k}$ is the wave vector of an incident particle and $\mathbf{P} = \hbar\mathbf{k}(1 + \beta'\hbar^2 k^2)$ is the momentum of a particle. The wave function of incident particle is

$$\psi_k(\mathbf{r}) = e^{i\mathbf{kr}}. \tag{19}$$

After scattering the motion of a particle is also free at long distances from the scatterer with the momentum $\mathbf{P}' = \hbar\mathbf{k}'(1 + \beta'\hbar^2 k'^2)$. As was noted above we consider elastic scattering so we have $k' = k$.

Equation (17) can be represented as follows

$$(\nabla^2 - \beta'\hbar^2\nabla^4 + k^2[1 + \beta'\hbar^2 k^2])\Psi = \frac{2m}{\hbar^2}U(\mathbf{r}, \mathbf{p})\Psi \tag{20}$$

and as was noticed earlier $\mathbf{p} = -i\hbar\nabla$.

The formal solution of equation (20) can be written as

$$\Psi(\mathbf{r}) = \psi_k(\mathbf{r}) + \int G(\mathbf{r}, \mathbf{r}')\frac{2m}{\hbar^2}U(\mathbf{r}', \mathbf{p}')\Psi(\mathbf{r}')\mathrm{d}\mathbf{r}', \tag{21}$$

that in fact is the integral equation and $G(\mathbf{r}, \mathbf{r}')$ is the Green's function which satisfies the following equation

$$(\nabla^2 - \beta'\hbar^2\nabla^4 + k^2[1 + \beta'\hbar^2 k^2])G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \tag{22}$$

The solution for the Green's function reads

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r} - \mathbf{r}') = \frac{1}{(2\pi)^3}\int \frac{e^{i\mathbf{q}(\mathbf{r}-\mathbf{r}')}}{k^2(1 + \beta'\hbar^2 k^2) - q^2(1 + \beta'\hbar^2 q^2)}\mathrm{d}\mathbf{q}. \tag{23}$$

After integration by angular variables we have

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4i\pi^2|\mathbf{r} - \mathbf{r}'|}\int_{-\infty}^{+\infty} \frac{q e^{iq|\mathbf{r}-\mathbf{r}'|}}{k^2(1 + \beta'\hbar^2 k^2) - q^2(1 + \beta'\hbar^2 q^2)}\mathrm{d}q. \tag{24}$$

The last integral can be calculated using the calculus of residues. Therefore, it is necessary to determine the contour of integration in the complex plane. So the way of enclosing the poles $q = \pm k$ might be defined. The manner of enclosing the poles can be determined from the boundary conditions imposed on the Green's function $G(|\mathbf{r} - \mathbf{r}'|)$ when $|\mathbf{r} - \mathbf{r}'| \to 0$. For obtaining the solution corresponding to the outgoing wave we choose the contour of integration similarly as we select the contour for the outgoing wave in ordinary quantum mechanics. Then it is easy to calculate integral (24) and we have

$$G(|\mathbf{r} - \mathbf{r}'|) = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|(1 + 2\beta'\hbar^2 k^2)}\left(e^{ik|\mathbf{r}-\mathbf{r}'|} - e^{-\sqrt{k^2 + 1/(\beta'\hbar^2)}|\mathbf{r}-\mathbf{r}'|}\right). \tag{25}$$

Green's function for a free particle in the deformed case is somewhat different than in the ordinary one. We have an additional multiplier $1/(1 + 2\beta'\hbar^2 k^2)$ tending to unity when the $\beta' \to 0$. Then in the deformed case we also have the additional function $e^{-\sqrt{k^2+1/(\beta'\hbar^2)}|\mathbf{r}-\mathbf{r}'|}$ rapidly decreasing with increasing $|\mathbf{r} - \mathbf{r}'|$. The last function tends to zero when $\beta' \to 0$. Since we want to find the asymptotic behaviour for the Green's function taking place when the difference $|\mathbf{r} - \mathbf{r}'|$ is large we may neglect the last function. Thus the Green's function at large distances from the scattering center reads

$$G(|\mathbf{r} - \mathbf{r}'|) = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|(1 + 2\beta'\hbar^2 k^2)}e^{ik|\mathbf{r}-\mathbf{r}'|}. \tag{26}$$

Using this Green's function we can rewrite integral equation (21) in the form

$$\Psi(\mathbf{r}) = \psi_k(\mathbf{r}) - \frac{m}{2\pi\hbar^2(1+2\beta'\hbar^2k^2)}\int \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}U(\mathbf{r}',\mathbf{p}')\Psi(\mathbf{r}')\mathrm{d}\mathbf{r}'. \tag{27}$$

As was noted earlier we want to obtain the solution of integral equation (27) at large distances from the center of scatter. The action of the potential energy operator $U(\mathbf{r}',\mathbf{p}')$ on the wave function makes a considerable contribution to integral of equation (27) in the bounded domain. The action of the operator $U(\mathbf{r}',\mathbf{p}')$ on the wave function $\Psi(\mathbf{r}')$ makes a negligibly small contribution to integral (27) for larger r'. So the actual values of the integration variable $\mathbf{r}'$ are bounded by some effective radius of the potential energy. When $r \to \infty$ the ratio r'/r is small and one can use the following development

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 - 2\mathbf{r}\mathbf{r}' + r'^2} \simeq r\left(1 - \frac{\mathbf{r}\mathbf{r}'}{r^2} + \cdots\right) = r - \mathbf{n}\mathbf{r}' + \cdots, \tag{28}$$

where $\mathbf{n} = \mathbf{r}/r$ is the unit vector along the direction of motion of the scattered particle.

Then we substitute decomposition (28) in equation (27) and taking into account the leading term of the asymptotic we represent equation (27) in the form

$$\Psi(\mathbf{r}) = \psi_k(\mathbf{r}) - \frac{m}{2\pi\hbar^2(1+2\beta'\hbar^2k^2)}\frac{e^{ikr}}{r}\int e^{-i\mathbf{k}'\mathbf{r}'}U(\mathbf{r}',\mathbf{p}')\Psi(\mathbf{r}')\mathrm{d}\mathbf{r}', \tag{29}$$

where $\mathbf{k}' = k\mathbf{n}$.

Expression (29) can be rewritten as follows

$$\Psi(\mathbf{r}) = e^{i\mathbf{k}\mathbf{r}} + \frac{e^{ikr}}{r}f, \tag{30}$$

where

$$f = -\frac{m}{2\pi\hbar^2(1+2\beta'\hbar^2k^2)}\int e^{-i\mathbf{k}'\mathbf{r}'}U(\mathbf{r}',\mathbf{p}')\Psi(\mathbf{r}')\mathrm{d}\mathbf{r}' \tag{31}$$

is the scattering amplitude.

Expression (30) formally coincides with the wave function of the scattering problem in ordinary quantum mechanics. The main difference between them is caused by different expressions for the scattering amplitude. As in the ordinary case the second term of the wave function (30) corresponds to the wave function of the scattered particle

$$\psi_{\text{scatt}} = \frac{e^{ikr}}{r}f. \tag{32}$$

The central problem of the scattering theory is the calculation of the differential cross-section. We define the differential cross-section similarly as it was done in the ordinary quantum mechanics

$$d\sigma = \frac{\mathbf{j}_{\text{scatt}}d\mathbf{S}}{j_0}, \tag{33}$$

where j_0 is the absolute value of the current density for the incident particles, $\mathbf{j}_{scatt}$ is the current density for the scattered particles and $d\mathbf{S}$ is the element of the area along the direction of motion for the scattered particles. The element of the area can be rewritten in the form $d\mathbf{S} = \mathbf{n}dS$ where $\mathbf{n} = \mathbf{r}/r$. After introducing a spatial angle $d\Omega = dS/r^2$ the differential cross-section can be represented in the form

$$\frac{d\sigma}{d\Omega} = \frac{(\mathbf{j}_{\text{scatt}}\mathbf{n})r^2}{j_0}. \tag{34}$$

For the calculation of the differential cross-section it is necessary to calculate the current density for the incident particles and the current density for the scattered particles. Substituting (19) into (16) we obtain

$$\mathbf{j}_0 = \frac{\hbar\mathbf{k}}{m}(1+2\beta'\hbar^2k^2). \tag{35}$$

The expression for the current density of incident particles is somewhat different than in the ordinary case. We have the additional factor $(1+2\beta'\hbar^2k^2)$ tending to unity if $\beta' \to 0$.

Substituting function (32) in relation (16) and after simple transformations we obtain the following expression for the current density for the outgoing particles

$$\mathbf{j}_{\text{scatt}} = \frac{\hbar\mathbf{k}}{m}\frac{|f|^2}{r^2}(1+2\beta'\hbar^2k^2). \tag{36}$$

Then we substitute relations (36) and (35) in expression (34) and obtain

$$\frac{d\sigma}{d\Omega} = |f|^2. \tag{37}$$

As we see the relation for the differential cross-section formally coincides with the expression for the cross-section in ordinary quantum mechanics, but the scattering amplitude is defined by relation (31) which, as we noted before, is somewhat different from the ordinary one.

4 Born approximation

The Born approximation can be used for finding the scattering amplitude in the deformed case. In the Born approximation we consider the potential energy $U(\mathbf{r},\mathbf{p})$ as a small perturbation and the integral equation (29) can be solved by the method of successive approximation. In the first approximation we substitute the plane wave (19) in expression (31) and taking into account the explicit form for the Yukawa potential (10) we obtain scattering amplitude

$$f_{\text{Yukawa}} = \frac{m}{2\pi\hbar^2(1+2\beta'\hbar^2k^2)}\int e^{-i\mathbf{k}'\mathbf{r}}e^2\left[\frac{e^{-\lambda\sqrt{r^2+b^2}}}{\sqrt{r^2+b^2}} + \frac{\hbar^2(2\beta-\beta')}{4}e^{-\lambda r}\left(\frac{1}{r}\nabla^2+\nabla^2\frac{1}{r}\right) + \frac{\hbar^2(2\beta-\beta')}{4}\frac{e^{-\lambda r}}{r}\times\right.$$
$$\left.\left(\frac{2}{3}\lambda^3 r - \lambda^2((\mathbf{r}\nabla)+(\nabla\mathbf{r})) + \lambda(r\nabla^2+\nabla^2 r)\right) + \frac{\hbar^2(2\beta-\beta')}{4}e^{-\lambda r}\left(\frac{2\lambda^2}{r} - 2\lambda\frac{1}{r}((\mathbf{r}\nabla)+(\nabla\mathbf{r}))\frac{1}{r}\right)\right]e^{i\mathbf{k}\mathbf{r}}d\mathbf{r}. \tag{38}$$

Then we calculate the contribution into the scattering amplitude caused by the term $\frac{e^{-\lambda\sqrt{r^2+b^2}}}{\sqrt{r^2+b^2}}$

$$\int e^{-i\mathbf{k}'\mathbf{r}}e^2\frac{e^{-\lambda\sqrt{r^2+b^2}}}{\sqrt{r^2+b^2}}e^{i\mathbf{k}\mathbf{r}}d\mathbf{r} = \frac{4\pi e^2 b}{\sqrt{\lambda^2+q^2}}K_1(b\sqrt{\lambda^2+q^2}), \tag{39}$$

where K_1 is the Bessel function [23] and

$$q = |\mathbf{k}'-\mathbf{k}| = 2k\sin\frac{\vartheta}{2}, \tag{40}$$

ϑ is the scattering amplitude.

We develop the Bessel function into the series and take into account only the first order terms in b. So contribution (39) can be represented in the form

$$\frac{4\pi e^2 b}{\sqrt{\lambda^2+q^2}}K_1(b\sqrt{\lambda^2+q^2}) \simeq \frac{4\pi e^2 b}{\sqrt{\lambda^2+q^2}}\left(\frac{1}{b\sqrt{\lambda^2+q^2}} + \frac{b\sqrt{\lambda^2+q^2}}{2}\left(\ln\left(\frac{b\sqrt{\lambda^2+q^2}}{2}\right)+\gamma-\frac{1}{2}\right)\right) =$$
$$= \frac{4\pi e^2}{\lambda^2+q^2} + 2\pi e^2 b^2\left(\ln\left(\frac{b\sqrt{\lambda^2+q^2}}{2}\right)+\gamma-\frac{1}{2}\right), \tag{41}$$

where $\gamma = 0.57721...$ is the Euler constant.

It is easy to calculate the contributions caused by another terms in integral (38). So we can write the scattering amplitude for the Yukawa potential taking into account the explicit form for the parameter $b = \hbar\sqrt{2\beta-\beta'}$

$$f_{\text{Yukawa}} = \frac{me^2}{2\pi\hbar^2(1+2\beta'\hbar^2k^2)}\left(\frac{4\pi}{\lambda^2+q^2} + \pi\hbar^2(2\beta-\beta')\left(\ln\left(\frac{\hbar^2(2\beta-\beta')(\lambda^2+q^2)}{4}\right)+2\gamma-1\right) -\right.$$
$$\left. 2\pi\hbar^2(2\beta-\beta')\left[\frac{k^2}{\lambda^2+q^2} + \frac{\lambda^2}{(\lambda^2+q^2)^2}\left(2k^2-\frac{\lambda^2}{3}\right)\right]\right). \tag{42}$$

Since we take into consideration only the first order terms in β, β' we can rewrite the last expression in the form

$$f_{\text{Yukawa}} = \frac{2me^2}{\hbar^2(\lambda^2+q^2)} + \frac{me^2}{2}(2\beta-\beta')\left[\ln\left(\frac{\hbar^2(2\beta-\beta')(\lambda^2+q^2)}{4}\right)+\right.$$
$$\left. 2\gamma - 1 - \frac{2k^2}{\lambda^2+q^2} - \frac{2\lambda^2}{(\lambda^2+q^2)^2}\left(2k^2-\frac{\lambda^2}{3}\right)\right] - \beta'\frac{4me^2k^2}{\lambda^2+q^2}. \tag{43}$$

Using the scattering amplitudes for the Yukawa potential (43) and taking into account relation (40) we calculate the differential cross-sections

$$\frac{d\sigma}{d\Omega} = \frac{4m^2e^4}{\hbar^4\left(\lambda^2+4k^2\sin^2\frac{\vartheta}{2}\right)^2} + \frac{4me^2}{\hbar^2\left(\lambda^2+4k^2\sin^2\frac{\vartheta}{2}\right)}\left(\frac{me^2}{2}(2\beta-\beta')\left[\ln\left(\frac{\hbar^2(2\beta-\beta')\left(\lambda^2+4k^2\sin^2\frac{\vartheta}{2}\right)}{4}\right)+\right.\right.$$
$$\left.\left. 2\gamma-1-\frac{2k^2}{\lambda^2+4k^2\sin^2\frac{\vartheta}{2}} - \frac{2\lambda^2}{\left(\lambda^2+4k^2\sin^2\frac{\vartheta}{2}\right)^2}\left(2k^2-\frac{\lambda^2}{3}\right)\right] - \beta'\frac{4me^2k^2}{\lambda^2+4k^2\sin^2\frac{\vartheta}{2}}\right). \tag{44}$$

Putting $\lambda = 0$ we obtain the differential cross-section for the Coulomb potential

$$\frac{d\sigma}{d\Omega} = \frac{m^2e^4}{4\hbar^4k^4\sin^4\frac{\vartheta}{2}} + \frac{me^2}{\hbar^2k^2\sin^2\frac{\vartheta}{2}}\left(\frac{me^2}{2}(2\beta-\beta')\left[\ln\left(\hbar^2(2\beta-\beta')k^2\sin^2\frac{\vartheta}{2}\right) + 2\gamma - 1 - \frac{1}{2\sin^2\frac{\vartheta}{2}}\right] - \beta'\frac{me^2}{\sin^2\frac{\vartheta}{2}}\right). \tag{45}$$

Similarly to [17, 18, 19] we introduce two parameters $\Delta x_{\min} = \hbar\sqrt{\beta+\beta'}$ (or $\xi = \Delta x_{\min}/a$) and $\eta = \frac{\beta}{\beta+\beta'}$ instead of β and β'. As was noted in [18] that our calculations hold if $2\beta - \beta' \geqslant 0$ and β, β' are nonnegative constants. These conditions lead to the constraints on the domain of variation for the dimensionless parameter η: $\frac{1}{3} \leqslant \eta \leqslant 1$. The last expression for the differential cross-section on the Coulomb potential can be rewritten using the parameters $\Delta x_{\min}$ and η as follows

$$\frac{d\sigma}{d\Omega} = \frac{m^2e^4}{4\hbar^4k^4\sin^4\frac{\vartheta}{2}}\left[1 + \zeta\left(\Delta x_{\min}, \eta, k, \vartheta\right)\right], \tag{46}$$

where

$$\zeta\left(\Delta x_{\min}, \eta, k, \vartheta\right) = 2\Delta x_{\min}^2 k^2(3\eta-1)\sin^2\tfrac{\vartheta}{2}\left[\ln\left(\Delta x_{\min}^2 k^2(3\eta-1)\sin^2\tfrac{\vartheta}{2}\right) + 2\gamma - 1\right] + \Delta x_{\min}^2 k^2(\eta-3) \tag{47}$$

is the specially introduced function which shows a correction to the differential cross-section caused by the deformation in terms of the wave vector.

We represent a relation for the differential cross-section (46) in terms of energy of the incident particle. We emphasize that in the deformed case kinetic energy is given by relation (18). Using (18) and taking into account only linear terms over deformation parameters we obtain

$$\frac{d\sigma}{d\Omega} = \frac{e^2}{16E^2\sin^4\frac{\vartheta}{2}}\left[1 + \delta(\Delta x_{\min}, \eta, E, \vartheta)\right], \tag{48}$$

where

$$\delta(\Delta x_{\min}, \eta, E, \vartheta) = \zeta(\Delta x_{\min}, \eta, E, \vartheta) + \tfrac{4m}{\hbar^2}\Delta x_{\min}^2 E(1-\eta) = \tfrac{4m}{\hbar^2}\Delta x_{\min}^2 E(3\eta-1)\sin^2\tfrac{\vartheta}{2}\times \left[\ln\left(\tfrac{2m}{\hbar^2}\Delta x_{\min}^2 E(3\eta-1)\sin^2\tfrac{\vartheta}{2}\right) + 2\gamma - 1\right] - \tfrac{2m}{\hbar^2}\Delta x_{\min}^2 E(\eta+1) \tag{49}$$

is the specially introduced function which similarly to (47) shows the corrections to the cross-section but expresses it in terms of energy.

Having expression (49) we can numerically estimate the correction to the cross-section caused by the minimal length effects. Relation (49) shows that function δ depends on the four variables $\Delta x_{\min}$, η, E and ϑ.

For the minimal length we use the upper bound obtained in [17, 18, 19]. As was shown in these works the upper bound for the minimal length is of the order $10^{-16} \div 10^{-17}$m. For the calculating of δ we take $\Delta x_{\min} = 10^{-16}$m. The parameter η have an arbitrary value from its domain of variation.

Fig.1 shows the dependence of δ on the scattering angle for different η and the energies of incident particles. We have three graphs and each of them corresponds to the given energy of incident particles. The function δ is negative for these energies and this means that the differential cross-section in the deformed case is smaller than in ordinary quantum mechanics. We emphasize that in the special case $\eta = \frac{1}{3}$ the function δ does not depend on the scattering angle and is fully determined by the energy of particle. When $\eta \neq \frac{1}{3}$ the function δ decreases with the increasing of the scattering angle.

5 Discussion

We studied the elastic scattering problem for the Yukawa and Coulomb potentials in the space with deformed Heisenberg algebra leading to nonzero minimal length. Using the shifted expansion over the deformation parameters β and β' we found the continuity equation in linear approximation over these parameters. The explicit expression for the density current contains new terms (in comparison with the undeformed case) caused by deformation. The terms proportional to β' are caused by kinetic energy in deformed space and the terms proportional to $2\beta - \beta'$ are caused by potential energy. It is necessary to stress that in contrast to the ordinary quantum mechanics in the case of deformation the potential energy also makes a contribution to the expression for the density current (see eq. (14),(15)). For the free particle the expression for the density current contains only an additional term proportional to β' (see eq. (16)).

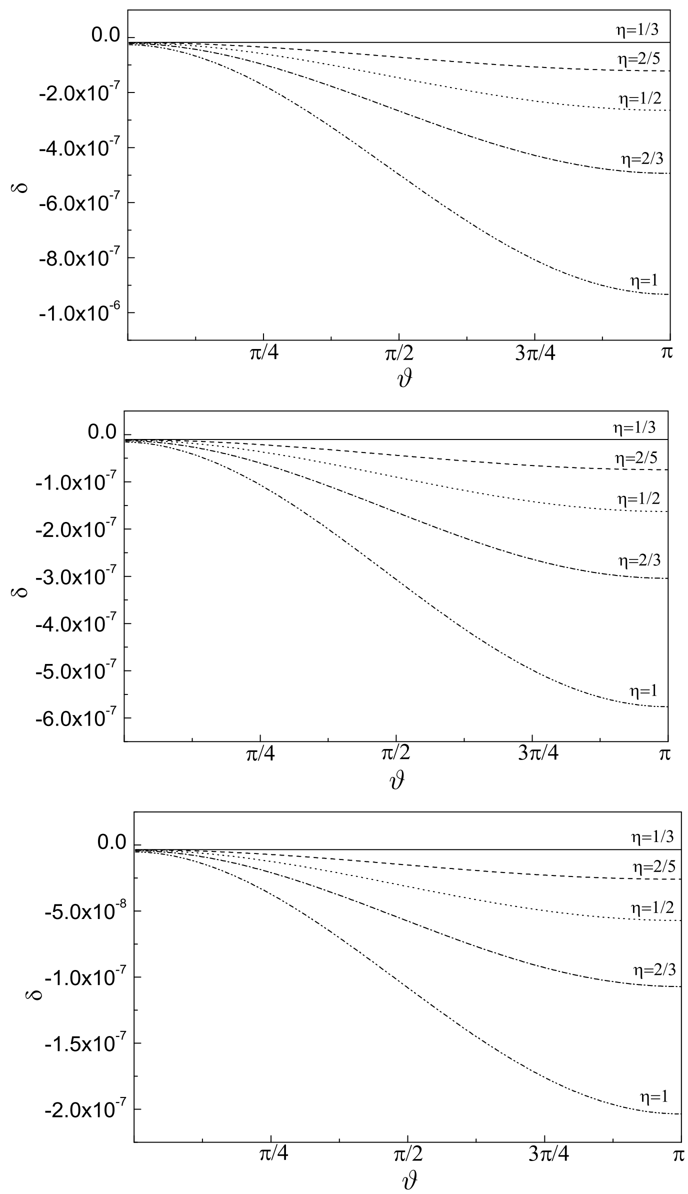

Figure 1: The dependence of $\delta(\Delta x_{\min}, E, \eta, \vartheta)$ on the the scattering angle ϑ for different η and energy E of the incident particles. The upper graph corresponds to the energy of the incident particles $E = 50$keV. The middle graph corresponds to the energy $E = 30$keV and the lower graph corresponds to the energy $E = 10$keV.

We obtained the Green's function for the free particle in the deformed space. Using this Green's function we obtained the wave function of the scattering problem in the deformed space. Similarly to the ordinary quantum mechanics the wave function consists of two terms. One of them is the plane wave corresponding to the wave function of the incident free particle and the second is the divergent spherical wave which corresponds to the scattered outgoing particle. We also obtained the relation for the scattering amplitude in the deformed space. As was shown the expression for the scattering amplitude in the deformed case is similar to the ordinary one but with an additional multiplier $1/(1+2\beta'\hbar^2k^2)$ tending to the unity when $\beta' \to 0$. Using the relation for the scattering amplitude we found the expression for the differential cross-section.

We calculated the scattering amplitude for the Yukawa potential in the Born approximation. Using this expression we found the differential cross-section for the Yukawa and Coulomb potentials in the Born approximation. Then we numerically estimate the corrections to the cross-section for the Coulomb (as a limit of the Yukawa one) potential caused by the minimal length effects. For calculations we used the estimations of minimal length obtained in the works [17, 18, 19]. We revealed that in the deformed case the differential cross-section is smaller than in the ordinary one. Absolute value of corrections to the cross-section caused deformation increases with the increasing of energy and scattering angle.

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